Ann. Nucl. Energy, **28**, 1193 (2001) (**revised**)

HGPT BASED SENSITIVITY METHODS FOR THE ANALYSIS OF SUBCRITICAL SYSTEMS

A. Gandini
University of Rome, Nuclear Engineering Department (DINCE)
Corso Vittorio Emanuele II 224, 00186 Rome - Italy

Abstract

In recent years an increasing interest is observed with respect to subcritical, accelerator driven systems (ADS). Considering the attention being given to these systems for their supposed ability to play a major role as actinides incinerators, as well as power production plants, the application of the heuristically-based generalized perturbation theory (HGPT) methodology for the cycle life analysis of these systems is reviewed and commented. It is discussed in particular the role of the importance function associated with the power control, and the definition of the concept of "generalized reactivity", merging into the standard concept of reactivity with the system approaching criticality. Basing on these results, a formulation is also described of a point kinetic equation, with physically significant coefficients, similar to those presented by Usachev (1955) using the standard adjoint flux as weighting function.

1. Introduction

Since the beginning of nuclear reactor physics studies, perturbation theory has played an important role. As well known, it was first proposed in 1945 by Wigner to study fundamental quantities such as the reactivity worths of different materials in the reactor core. It is also well known that this first formulation, today widely used by reactor analysts, makes a consistent use of the adjoint flux concept. The advantage of using perturbation theory lies in the fact that instead of making a new, often lengthy direct calculation of the eigenvalue (and then of the real flux) for every perturbed system configuration, a simple integration operation is required in terms of unperturbed quantities.

It is interesting that as early as 1948 Soodak associated to the adjoint flux the concept of importance, viewing it as proportional to the contribution of a neutron, inserted in a given point of a critical system, to the asymptotic power.

Along with the introduction of the concept of importance and, parallel to it, along with the development of calculation methods and machines, from the early 60' a flourishing of perturbation

methods, at first in the linear domain and then in the nonlinear one, have been proposed for analysis of reactor core physics, shielding, thermohydraulics, as well as other fields.

The perturbation formulations proposed by various authors may be subdivided into three main categories, according to the approach followed in their derivation:

1. The heuristic approach, making exclusive use of importance conservation concepts, adopted first by Usachev (1963) and then extensively developed by Gandini (1967-1987). It will be referred to, in the following, as heuristic generalized perturbation theory (HGPT) method.
2. The variational approach adopted, in particular, by Lewins (1965), Pomraning (1967), Stacey (1976), Harris and Becker (1976) and Williams (1979).
3. The differential method, proposed by Oblow (1976) and extensively developed by Cacuci (1980), based on a formal differentiation of the response considered.

Each of the above methods has its own merit, although all of them can be shown equivalent to each other (Greenspan, 1975).

In this review we shall discuss the potential applications of the HGPT methodology to the analysis of subcritical systems .

A first indication of the potential use of the HGPT methodology with respect to neutron kinetic analysis of critical and noncritical systems (with an external source) and to the possibility of analyzing integral experiments in reactor facilities at subcritical conditions was suggested in 1968 (Gandini). In particular, the neutron and precursor importances associated with a given response was considered. In subsequent articles (Gandini, 1976, 1981), the use of HGPT methods for time-dependent problems was again discussed. In particular, the composite neutron, precursor and multi-channel temperature field, generally in presence of external neutron and enthalpy sources, was suggested for application of the HGPT methodology in dynamic studies.

Considering the increasing attention being given to the subcritical, accelerator driven systems (ADS) for their supposed ability to play a major role as actinides incinerators, as well as power production plants, the application of the HGPT methodology for the cycle life analysis of these systems (Gandini, 1997) was proposed in 1997 basing on a previous procedure (Gandini 1987, 1988) developed for critical ones. In the present paper, we shall shortly review and comment these works. In particular the role is discussed of the importance function associated with the power control, and the definition of the concept of "generalized reactivity", merging into the standard concept of reactivity with the system approaching criticality. Basing on these results, a formulation is finally described of a point kinetic equation, with physically significant coefficients, similar to that presented by Usachev (1955) using the standard adjoint flux as weighting function and basing on a previous work by Hurwitz (1949).

2. The HGPT method

In the HGPT method the importance function is uniquely defined in relation to a given system response, for example, a neutron dose, the quantity of plutonium in the core at end of cycle, the temperature of the outlet coolant.

The HGPT method was first derived in relation to the linear neutron density field. Then it was extended to other linear ones. For all these fields the equation governing the importance function was obtained directly by imposing that on average the contribution to the chosen response from a particle [a neutron, or a nuclide, or an energy carrier] introduced at a given time in a given phase space point of

the system is conserved through time (importance conservation principle). Obviously such importance will result generally dependent on the time, position, and, when the case, energy and direction, of the inserted particle.

Consider a particle field density represented by vector **f** governed by equation

$$\mathbf{m}(\mathbf{f}|p) = 0 \ , \tag{2.1}$$

p_j (j=1,2,...) representing system parameters, and a response Q of the type[+]

$$Q = \int_{t_o}^{t_F} < \mathbf{h}^{+T}\mathbf{f} > dt \equiv << \mathbf{h}^{+}, \mathbf{f} >> \ , \tag{2.2}$$

where $\mathbf{h}^+$ is an assigned vector function and where < > indicate integration over the phase space.
Along with the HGPT methodology (Gandini, 1987), the importance function (**f***) obeys the equation

$$H^*\mathbf{f}^* + \mathbf{h}^{+} = 0 \tag{2.3}$$

where H* is the operator obtained by reversing the Jacobian operator H ($\equiv \partial\mathbf{m}/\partial\mathbf{f}$), this implying transposing matrix elements, changing sign of odd derivatives, inverting the order of the operators. The sensitivities s_j with rrespect to the system parameters result

$$s_j \equiv \frac{dQ}{dp_j} = << \frac{\partial \mathbf{h}^{+}}{\partial p_j}, \mathbf{f} >> + << \mathbf{f}^{*}, \frac{\partial \mathbf{m}}{\partial p_j} >> \ . \tag{2.4}$$

The HGPT method was extended to any field governed by linear operators for which the rules for their reversal were known. In Appendix A the derivation of the governing equations relevant to the neutron and nuclide densities in a critical nuclear reactor system is illustrated.

3. Source driven systems

The methodology described in Appendix A for long term nuclide/neutron core cycle evolution analysis may be very well applied to source driven, subcritical systems.

One of the advantages often claimed for the subcritical source driven power systems is associated to the fact that the power level may be basically controlled by the source strength (via the regulation of the accelerator current). So, no control, or regulating elements would be necessary, if a sufficient breeding is available (and/or an appropriate core burnable poison distribution is provided at the beginning of cycle) in the core for compensating the reactivity loss during burnup. To represent this,

---

[+] Expression (1) is also representative of more general responses, of the type Q=<<L(**f**)>>, L being a given function of **f**. In fact, if we extend **f** to the field $\hat{\mathbf{f}} = \begin{vmatrix} \mathbf{f} \\ y \end{vmatrix}$, where y=L(**f**), Q reduces to the form of Eq. (2.2), i.e., $Q = << \mathbf{h}^{+}, \mathbf{f} >>$, having set $\mathbf{h}^{+} = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$.

we shall rewrite Eqs. (A.1), (A.2) and (A.3), relevant to the neutron density **n**, the nuclide density **c** and the control function ρ, in the form

$$\mathbf{m}_{(n)}(\mathbf{n},\mathbf{c},\rho \mid \mathbf{p}) = -\frac{\partial \mathbf{n}}{\partial t} + B\mathbf{n} + \rho \mathbf{s}_n = 0 \tag{3.1}$$

$$\mathbf{m}_{(c)}(\mathbf{n},\mathbf{c} \mid \mathbf{p}) = -\frac{\partial \mathbf{c}}{\partial t} + E\mathbf{c} + \mathbf{s}_c = 0 \tag{3.2}$$

$$\mathbf{m}_{(\rho)}(\mathbf{n},\mathbf{c} \mid \mathbf{p}) = <\mathbf{c}, S\mathbf{n}> - W = 0 \tag{3.3}$$

where B and E depend on fuel and neutron densities **c** and **n**, respectively.

Since we generally consider systems at quasi-static, i.e., stationary conditions, the time derivative at second member of Eq. (3.1) may be neglected in the course of the integration process.

Any response, functional of variables **n**, **c**, and ρ, could be considered for analysis. We think instructive to limit here consideration to the response defined by the expression

$$Q = \rho(t_F) \equiv \int_{t_o}^{t_F} \delta(t - t_F)\rho(t)dt \tag{3.4}$$

which corresponds to the relative source strength required at t_F to assure the power level imposed. We may assume that, at unperturbed conditions, ρ (t)=1 in the interval (t_o,t_F). If some system parameter (for instance, the initial enrichment, or some other material density) is altered, as in an optimization search analysis, it may be of interest to evaluate the corresponding change of ρ at the end of cycle, to make sure that given upper limit specifications of the source strength are non exceeded.

Along with the HGPT methodology, the equations for the corresponding importance functions result

$$-\frac{\partial \mathbf{n}^*}{\partial t} = B^*\mathbf{n}^* + \Omega_c^*\mathbf{c}^* + S^T\mathbf{c}\rho^* \tag{3.5}$$

$$-\frac{\partial \mathbf{c}^*}{\partial t} = E^*\mathbf{n}^* + \Omega_n^*\mathbf{n}^* + S\mathbf{n}\rho^* \tag{3.6}$$

$$<\mathbf{n}^*, \mathbf{s}_n> + \delta(t-t_F) = 0 \tag{3.7}$$

Ω_c^* and Ω_n^* being coupling operators defined with Eq. (A.8).

Eq. (3.7) corresponds to an orthonormal condition for $\mathbf{n}^*$.

In order to determine the 'final' value $\mathbf{n}^*(t_F)$ required for starting the integration of Eq. (3.5), in consideration of the nature of the above governing equations, we shall first write $\mathbf{n}^*$ and ρ^* in the form[+]

$$\mathbf{n}^*(\mathbf{r},t) = \mathbf{n}_F^*\,\delta(t - t_F) + \tilde{\mathbf{n}}^*(\mathbf{r},t) \tag{3.8}$$

$$\rho^*(t) = \rho_F^*\,\delta(t - t_F) + \tilde{\rho}^*(t) \tag{3.9}$$

with $\tilde{\mathbf{n}}^*(\mathbf{r},t)$ and $\tilde{\rho}^*(t)$ being finite functions, vanishing at t_F.

Replacing into Eq. (3.5), integrating in the interval $(t_F - \varepsilon,\ t_F + \varepsilon)$, and then making $\varepsilon \to 0$, we obtain the equation

$$B^*\mathbf{n}_F^* + S^T\mathbf{c}(t_F)\rho_F^* = 0 \tag{3.10}$$

Let us now define $\bar{\mathbf{n}}_F^*$ as obeying equation

$$B^*\bar{\mathbf{n}}_F^* + S^T\mathbf{c}(t_F) = 0 \tag{3.11}$$

We note that $\bar{\mathbf{n}}_F^*$ corresponds to the importance relevant to functional $\langle \mathbf{c}(t_F), S\mathbf{n}(t_F)\rangle$, i.e., to the system power W. From the source reciprocity relationship (Section 2), we may write

$$\langle \mathbf{c}(t_F),\, S\mathbf{n}(t_F)\rangle = \langle \bar{\mathbf{n}}_F^*,\, \mathbf{s}_n\rangle = W\,. \tag{3.12}$$

From constraint, Eq. (3.7), we easily obtain

$$\rho_F^* = -\frac{1}{\langle \bar{\mathbf{n}}_F^*, \mathbf{s}_n \rangle} = -\frac{1}{W} \tag{3.13}$$

and then

$$\mathbf{n}_F^* = \bar{\mathbf{n}}_F^*\,\rho_F^* = -\frac{\bar{\mathbf{n}}_F^*}{W}\,. \tag{3.14}$$

From this 'final' value, a recurrent calculation scheme may be defined starting from t_F and proceeding backward.

Along the HGPT methodology, the sensitivity coefficient relevant to the k'th parameter p_k is found as

---

[+] The diverging of $\mathbf{n}^*(\mathbf{r},t)$ at t_F may be explained on physical grounds recalling the meaning of importance (in this case, the contribution to the given response by a neutron with the same space/time coordinates) and considering that the response here is $\rho(t_F)$, i.e., the control assumed to maintain the power at a prefixed level. A neutron introduced at t_F into the system would in fact produce a (delta-like) impulse of control ρ to balance its effect on the power level. Then, the importance associated to such neutron would be characterized by a similar delta-like behavior. A quite similar reasoning applies in relation to the diverging of importance $\rho^*(t)$ at t_F, considering that its physical meaning corresponds to the contribution to the response [defined as $\rho(t_F)$] due to a unit energy insertion at t_F or, which is the same, to an overall power pulse $\delta(t-t_F)$.

$$\frac{\partial \rho(t_F)}{\partial p_k} = \rho_F^* [< \bar{\mathbf{n}}_F^* \frac{\partial}{\partial p_k}(B\mathbf{n} + \mathbf{s}_n) > + \frac{\partial}{\partial p_k}(< \mathbf{c}, S\mathbf{n} > - W)]_{t_F}$$
$$+ \int_{t_o}^{t_F} [< \tilde{\mathbf{n}}^*, \frac{\partial}{\partial p_k}(B\mathbf{n} + \mathbf{s}_n) + < \mathbf{c}^*, \frac{\partial E}{\partial p_k}\mathbf{c} > + \tilde{\rho}^* \frac{\partial}{\partial p_k}(< \mathbf{c}, S\mathbf{n} > - W)]dt \quad (3.15)$$

with ρ_F^* given by Eq. (3.13). The first term at right side accounts for effects on $\rho(t_F)$ due to parameter changes at t_F, in particular, if $p_k \equiv W$, it gives the (trivial) result $\frac{\partial \rho(t_F)}{\partial W} = \frac{1}{W}$. The second, integral term accounts for analogous effects on $\rho(t_F)$ produced by parameter changes at times $t < t_F$.

Rather than on the source term, a control on the neutron absorption in the multiplying region could be of interest. In this case, the (intensive) control variable ρ would represent the average penetration of the control elements, or the average density of the soluble boron in the coolant, and then would enter into the (transport, or diffusion) operator B. The orthonormal condition for the neutron importance $\mathbf{n}^*$ would now be, rather than Eq. (3.7),

$$< \mathbf{n}^*, \frac{\partial B}{\partial \rho}\mathbf{n} > + \delta(t - t_F) = 0 \ . \quad (3.16)$$

In this case, the sensitivity coefficient with respect to a given parameter p_k would always be given by Eq. (3.15), with $\bar{\mathbf{n}}_F^*$ obeying Eq. (3.11), but with

$$\rho_F^* = -\frac{1}{< \bar{\mathbf{n}}_F^*, \frac{\partial B}{\partial \rho}\mathbf{n} >} . \quad (3.17)$$

In general, a control strategy, by which an automatic resetting of the imposed overall power is actuated, might imply a control intervention on both the neutron source strength and the absorbing elements within the multiplying region. In this case, ρ (which remains a unique, intensive control variable) would affect both operator *B* and the neutron source [in this latter case, via an appropriate ρ- and parameter dependent coefficient $\alpha(\rho \mid \mathbf{p})$, assumed unity at unperturbed conditions]. The distribution between these two control mechanisms could be described by appropriate parameters (subject to perturbation analysis). The sensitivity coefficient, in this case, with respect to a given parameter p_k would always be given by Eq. (3.15), with $\bar{\mathbf{n}}_F^*$ obeying Eq. (3.11), but with

$$\rho_F^* = -\frac{1}{< \bar{\mathbf{n}}_F^*, (\frac{\partial B}{\partial \rho}\mathbf{n} + \frac{\partial \alpha}{\partial \rho}\mathbf{s}_n) >} . \quad (3.18)$$

3.1. Stationary Case

To study a given subcritical system at stationary conditions (which may be interpreted at the beginning of its cycle life), we may consider the same system above in which the neutron source and the nuclide density are assumed time-independent during an arbitrary time interval (t_o,t_B). We assume that at t_o the neutron density ($\mathbf{n}_o$), as well as the control (ρ_o) have already reached stationary conditions. So, also these two quantities are time-independent in the same time interval. Their governing equations can then be written, in case the power level is controlled by the source strength,

$$B\mathbf{n}_o + \rho_o \mathbf{s}_{n,o} = 0 \tag{3.19}$$

$$<\mathbf{c}_o, S\mathbf{n}_o> - W_o = 0 \, . \tag{3.20}$$

Also here we shall assume that at unperturbed conditions $\rho_o = 1$.

The same equations derived previously are applicable to this case, with the advertence of replacing t_F with t_B and setting the coupling operators Ω_c^* and Ω_n^* appearing in Eqs. (3.5) and (3.6) equal to zero. The sensitivity coefficient of the response $\rho(t_B)$ [$=\rho(t)=\rho_o$, i.e., constant in the whole interval (t_o,t_B)] relevant to the j'th parameter p_k can then be obtained. Since in this case $\mathbf{c}^*$, as well as $\tilde{\mathbf{n}}^*(\mathbf{r},t)$ and $\tilde{\rho}^*(\mathbf{r},t)$ vanish, recalling Eq. (3.15), we obtain

$$\frac{\partial \rho_o}{\partial p_k} = \rho_o^* [< \mathbf{n}_o^*, \frac{\partial}{\partial p_k}(B\mathbf{n}_o + \mathbf{s}_{n,o}) > + \frac{\partial}{\partial p_k}(< \mathbf{c}_o, S\mathbf{n}_o > - W_o)] \tag{3.21}$$

where

$$\rho_o^* = -\frac{1}{W_o} \tag{3.22}$$

and $\mathbf{n}_o^*$ obeys equation

$$B^* \mathbf{n}_o^* + S^T \mathbf{c}_o = 0 \quad . \tag{3.23}$$

If, rather than via the source strength, the power level reset control is assumed to be regulated via neutron absorption, so that the control ρ_o would enter into operator B, the sensitivity coefficient would be given always by Eq. (3.21), but with

$$\rho_o^* = -\frac{1}{< \mathbf{n}_o^*, \frac{\partial B}{\partial \rho}\mathbf{n} >} \, . \tag{3.24}$$

We might as well consider a (fictitious) control mechanism affecting the fission source, rather than the neutron absorption, i.e., we might choose as control a coefficient multiplying the fission matrix (F) and, therefore, entering into the Boltzmann, or diffusion, operator B (=A+ρ_oF). The sensitivity coefficient would be given again by Eq. (3.21), but with

$$\rho_o^* = -\frac{1}{< \mathbf{n}_o^*, \mathrm{F}\mathbf{n}_o >} . \tag{3.25}$$

3.2. Reactivity of Subcritical Systems

For resetting the power level, we have considered above different control mechanisms to which the following types of equations governing the neutron density may be associated:

$$B(\mathbf{p})\mathbf{n}_o + \rho_o \mathbf{s}_{n,o}(\mathbf{p}) = 0 \qquad \text{(source control)} \tag{3.26}$$

$$B(\rho_o|\mathbf{p})\mathbf{n}_o + \mathbf{s}_{n,o}(\mathbf{p}) = 0 \qquad \text{(neutron absorption, or fission control)} \tag{3.27}$$

$$B(\rho_o|\mathbf{p})\mathbf{n}_o + \alpha(\rho_o|\mathbf{p})\mathbf{s}_{no}(\mathbf{p}) = 0 \qquad \text{(mixed control)}^{+} \tag{3.28}$$

where the control and parameter dependence is indicated. Coefficient α is given and reflects the mixed strategy chosen. Eqs. (3.26), (3.27) and (3.28) may be generally represented by equation

$$\mathbf{m}_{(n,o)}(\mathbf{n}_o, \rho_o|\mathbf{p}) = 0 . \tag{3.29}$$

The sensitivity expression (3.21) may be generalized so that

$$\frac{d\rho_o}{dp_j} = -\frac{< \mathbf{n}_o^*, \frac{\partial \mathbf{m}_{(n,o)}}{\partial p_j} > + \frac{\partial}{\partial p_j}(< \mathbf{c}_o, \mathrm{S}\mathbf{n}_o > - W_o)}{< \mathbf{n}_o^*, \frac{\partial \mathbf{m}_{(n,o)}}{\partial \rho_o} >} , \tag{3.30}$$

with $\mathbf{n}_o^*$ obeying Eq. (3.23).

A corresponding perturbation expression may now be obtained. Assuming that the power W_o appearing in Eq. (3.30) is not subject to perturbation, we may write:

[+] A mixed control strategy may be considered also using Eq. (3.26), or Eq. (3.27). Adopting, for instance, Eq. (3.26), relevant to the neutron source control, part of the power level would be taken care of parametrically (e.g., by properly changing the control rod position, or the soluble boron density). The remaining reset would be taken care of intrinsically, by the ρ-control chosen.

$$\delta\rho_o = -\frac{< \mathbf{n}_o^*, \delta\mathbf{m}_{(n,o)} > + < \mathbf{n}_o, \delta(S^T\mathbf{c}_o) >}{< \mathbf{n}_o^*, \dfrac{\partial\mathbf{m}_{(n,o)}}{\partial\rho_o} >}, \tag{3.31}$$

where $\delta\mathbf{m}_{(n,o)} = \sum_j \delta p_j \dfrac{\partial\mathbf{m}_{(n,o)}}{\partial p_j}$ and $\delta(S^T\mathbf{c}_o) = \sum_j \delta p_j \dfrac{\partial(S^T\mathbf{c}_o)}{\partial p_j}$.

As said previously, $\delta\rho_o$ corresponds to the control change necessary to reestablish the power level existing before the perturbation $\delta\mathbf{m}_{(n,o)}$. We may as well say that the perturbation $\delta\mathbf{m}_{(n,o)}$ [and $\delta(S^T\mathbf{c}_o)$] would produce a power level change equivalent to that produced by a control change δK_ρ given by the equation

$$\delta K_\rho = \frac{< \mathbf{n}_o^*, \delta\mathbf{m}_{(n,o)} > + < \mathbf{n}_o, \delta(S^T\mathbf{c}_o) >}{< \mathbf{n}_o^*, \dfrac{\partial\mathbf{m}_{(n,o)}}{\partial\rho_o} >}. \tag{3.32}$$

In the case of the (fictitious) control on the neutron fission, setting λ in place of ρ to distinguish this peculiar case, we may explicitly write

$$\delta K_\lambda = \frac{< \mathbf{n}_o^*, \delta B\mathbf{n}_o >}{< \mathbf{n}_o^*, F\mathbf{n}_o >} + \frac{< \mathbf{n}_o^*, \delta\mathbf{s}_{n,o} >}{< \mathbf{n}_o^*, F\mathbf{n}_o >} + \frac{< \mathbf{n}_o, \delta(S^T\mathbf{c}_o) >}{< \mathbf{n}_o^*, F\mathbf{n}_o >}. \tag{3.33}$$

The first term at the right side closely resembles the reactivity expression for critical systems[+]. So, we shall call a quantity δK_λ as given by expression (3.33) a 'generalized reactivity'. The second term may be defined the "source reactivity", whereas the last one a "direct effect". To account for a generic ρ-mode control mechanism, we shall extend this definition to δK_ρ, similarly defined by Eq. (3.32), i.e.,

$$\delta K_\rho = \frac{< \mathbf{n}_o^*, \delta B\mathbf{n}_o >}{< \mathbf{n}_o^*, \dfrac{\partial\mathbf{m}_{(n,o)}}{\partial\rho_o} >} + \frac{< \mathbf{n}_o^*, \delta\mathbf{s}_{n,o} >}{< \mathbf{n}_o^*, \dfrac{\partial\mathbf{m}_{(n,o)}}{\partial\rho_o} >} + \frac{< \mathbf{n}_o, \delta(S^T\mathbf{c}_o) >}{< \mathbf{n}_o^*, \dfrac{\partial\mathbf{m}_{(n,o)}}{\partial\rho_o} >}. \tag{3.34}$$

and call it generalized ρ-mode reactivity.

3.3. Point Kinetics

Let us now consider equations governing the neutron flux ϕ ($\equiv V\mathbf{n}$) and precursor m_i (i=1,2,...,I) in a multigroup (G groups) neutron energy scheme :

[+] The first term at right hand side of Eq. (3.33) can be demonstrated to formally approach the standard reactivity expression as the (reference) system considered gets close to criticality conditions (Gandini, 1997).

$$V^{-1}\frac{d\phi}{dt} = A\phi + (1-\beta)\chi_P S_f^G \phi + \chi_D \mathbf{u}\sum_{i=1}^{I}\lambda_i m_i + \mathbf{s}_n \tag{3.35}$$

$$\frac{dm_i}{dt} = \beta_i \nu\Sigma_f^T \phi - \lambda_i m_i \tag{3.36}$$

where A is the transport, capture and scattering matrix operator, V the diagonal neutron velocity matrix, **u** is a unit (G component) vector and

$$S_f^X = \begin{vmatrix} \nu\Sigma_{f,1} & \cdots & \nu\Sigma_{f,G} \\ \cdots & \cdots & \cdots \\ \nu\Sigma_{f,1} & \cdots & \nu\Sigma_{f,G} \end{vmatrix}_{(X \text{ rows})}, \quad \Sigma_f^T = \begin{vmatrix} \Sigma_{f,1} & \cdots & \Sigma_{f,G} \end{vmatrix}, \quad \chi_z = \text{diag} \begin{vmatrix} \chi_{z,1} & \cdots & \chi_{z,G} \end{vmatrix}$$

Setting

$$X_D = \begin{vmatrix} \chi_{D,1} & \cdots & \chi_{D,1} \\ \cdots & \cdots & \cdots \\ \chi_{D,G} & \cdots & \chi_{D,G} \end{vmatrix}_{(GxI)}, \quad \Lambda = \text{diag} \begin{vmatrix} \lambda_1 & \cdots & \lambda_I \end{vmatrix}, \quad B = \text{diag} \begin{vmatrix} \beta_1 & \cdots & \beta_I \end{vmatrix}$$

Eqs. (3.35) and (3.36) may be written

$$V^{-1}\frac{d\phi}{dt} = A\phi + (1-\beta)\chi_P S_f^G \phi + X_D \Lambda \mathbf{m} + \mathbf{s}_n \tag{3.37}$$

$$\frac{d\mathbf{m}}{dt} = BS_f^I \phi - \Lambda \mathbf{m} \tag{3.38}$$

or, in matrix form,

$$\frac{d}{dt}\begin{vmatrix} V^{-1}\phi \\ \mathbf{m} \end{vmatrix} = \begin{vmatrix} A + (1-\beta)\chi_P S_f^G + & X_D\Lambda \\ BS_f^I & -\Lambda \end{vmatrix} \begin{vmatrix} \phi \\ \mathbf{m} \end{vmatrix} + \begin{vmatrix} \mathbf{s}_n \\ 0 \end{vmatrix} \tag{3.39}$$

At unperturbed, steady state conditions Eq. (3.39) reduces to:

$$\begin{vmatrix} A_o + (1-\beta)\chi_P S_{f,o}^G & X_D\Lambda \\ BS_{f,o}^I & -\Lambda \end{vmatrix} \begin{vmatrix} \phi_i \\ \mathbf{m}_i \end{vmatrix} + \begin{vmatrix} \mathbf{s}_{n,i} \\ 0 \end{vmatrix} = 0 \tag{3.40}$$

or

$$A_o\phi_o + \left[\chi_P(1-\beta) + \chi_D\beta\right]\, S_{f,o}\phi_o + \mathbf{s}_n = 0 \tag{3.41}$$

Consider the neutron importance $\mathbf{n}^*_{s,o}$ associated to the source power control, as defined by Eq. (3.14), and the corresponding precursor density $\mathbf{m}^*_{s,o}$ (Gandini, 1976). These importances are governed by the equation

$$\begin{vmatrix} A^*_o + (1-\beta)S^{G,T}_{f,o}\chi_P & S^{I,T}_{f,o}B \\ \Lambda X^T_D & -\Lambda \end{vmatrix} \begin{vmatrix} \mathbf{n}^*_{s,o} \\ \mathbf{m}^*_{s,o} \end{vmatrix} + \begin{vmatrix} \frac{\gamma}{W_o}\Sigma_{f,o} \\ 0 \end{vmatrix} = 0 \tag{3.42}$$

γ being the number of energy units per fission and W_o the system power at stationary, unperturbed conditions.

We may also write:

$$A^*_o\mathbf{n}^*_{s,o} + \nu S^T_{f,o}\left[(1-\beta)\chi_P + \beta\chi_D\right]\, \mathbf{n}^*_{s,o} + \frac{\gamma}{W_o}\Sigma_{f,o} = 0 \tag{3.43}$$

Function c^*_i results, by definition of importance:

$$m^*_{s,i,o} \equiv m^*_{s,o} = \mathbf{u}^T\chi_D\mathbf{n}^*_{s,o} \tag{3.44}$$

Rewrite Eq. (3.39) in the form (writing S_f rather than S^G_f):

$$V^{-1}\frac{d\phi}{dt} = (A_o + \delta A)\phi + (1-\beta)\chi_P(S_{f,o} + \delta S_f)\phi + \chi_D\mathbf{u}\sum_{i=1}^{I}\lambda_i m_i + \mathbf{s}_n \tag{3.45}$$

$$\frac{dm_i}{dt} = \beta_i\nu\Sigma^T_f\phi - \lambda_i m_i \tag{3.46}$$

Multiplying Eqs. (3.45) and (3.46) on the left by $\mathbf{n}^{*T}_{s,o}$,and, $m^*_{s,o}$, respectively, space-integrating and recalling expression (3.44), we obtain

$$\frac{d<\mathbf{n}^*_{s,o}, V^{-1}\phi>}{dt} = <\mathbf{n}^*_{s,o}, [(A_o + \delta A) + [(1-\beta)\chi_P(S_{f,o} + \delta S_f)]\phi> + \sum_{i=1}^{I}\lambda_i <m^*_o m_i> + <\mathbf{n}^*_{s,o}, (\mathbf{s}_{n,o} + \delta\mathbf{s}_n)> \tag{3.47}$$

$$\frac{d<m^*_{s,o} m_i>}{dt} = \beta_i <m^*_{s,o}\nu\Sigma^T_f\phi> - \lambda_i <m^*_{s,o} c_i> \tag{3.48}$$

Recalling Eq. (3.43) governing the importance function $\mathbf{n}^*_{s,o}$ and the importance reciprocity relationship

$$\frac{\gamma}{W_o} < \Sigma_{f,o}, \phi_o > = < \mathbf{n}^*_{s,o}, \mathbf{s}_{n,o} > (=1), \tag{3.49}$$

adding and subtracting the term $\beta < \mathbf{n}^*_{s,o}, \chi_D S_f \phi >$ at the right side of Eq. (3.47), after some manipulations this transforms into

$$\begin{aligned} \frac{d < \mathbf{n}^*_{s,o}, V^{-1}\phi >}{dt} = & < \mathbf{n}^*_{s,o}, \left\{ \delta A + [(1-\beta)\chi_P + \beta\chi_D]\delta S_f \right\} \phi > + < \mathbf{n}^*_{s,o}, \delta \mathbf{s}_n > \\ & + \sum_{i=1}^{M} \lambda_i < m^*_{s,o} m_{i,o} > - \beta < \mathbf{n}^*_{s,o}, \chi_D S_f \phi > + 1 - \frac{W}{W_o} + \frac{\gamma}{W_o} < \delta\Sigma_f, \phi > \end{aligned} . \tag{3.50}$$

Let us define the source term

$$< \mathbf{n}^*_{s,o}, \bar{\chi} S_f \phi > \equiv (1-\beta) < \mathbf{n}^*_{s,o}, \chi_P S_f \phi > + \beta < \mathbf{n}^*_{s,o}, \chi_D S_f \phi > \tag{3.51}$$

and assume that

$$\begin{aligned} \frac{d < \mathbf{n}^*_{s,o}, V^{-1}\phi >}{dt} &= \frac{d}{dt}\left[\frac{< \mathbf{n}^*_{s,o}, V^{-1}\phi >}{< \mathbf{n}^*_{s,o}, \bar{\chi} S_f \phi >} \frac{< \mathbf{n}^*_{s,o}, \bar{\chi} S_f \phi >}{< \Sigma_f, \phi >} < \Sigma_f, \phi > \right] \\ &\cong \frac{d}{dt}\left[\frac{< \mathbf{n}^*_{s,o}, V^{-1}\phi_o >}{< \mathbf{n}^*_{s,o}, \bar{\chi} S_f \phi_o >} \frac{< \mathbf{n}^*_{s,o}, \bar{\chi} S_f \phi_o >}{< \Sigma_f, \phi_o >} < \Sigma_f, \phi > \right] \end{aligned} \tag{3.52}$$

If we define then the quantities:

$$P(t) = \frac{W(t)}{W_o} \qquad \text{(relative power)} \tag{3.53}$$

$$\ell_{eff} = \frac{< \mathbf{n}^*_{s,o}, V^{-1}\phi_o >}{< \mathbf{n}^*_{s,o}, \bar{\chi} S_f \phi_o >} \qquad \text{(effective prompt neutron lifetime)} \tag{3.54}$$

$$\rho_{gen} = \frac{< \mathbf{n}^*_{s,o}, \left\{ \delta A + (1-\beta)\chi_P + \beta\chi_D]\delta S_f \right\} \phi_o > + \frac{\gamma}{W_o} < \delta\Sigma_f, \phi_o >}{< \mathbf{n}^*_{s,o}, \bar{\chi} S_f \phi_o >} \qquad \text{(generalized reactivity)} \tag{3.55}$$

$$\rho_{source} = \frac{< \mathbf{n}^*_{s,o}, \delta \mathbf{s}_n >}{< \mathbf{n}^*_{s,o}, \bar{\chi} S_{f,o} \phi_o >} \qquad \text{(source reactivity)} \tag{3.56}$$

$$\alpha = \frac{< \mathbf{n}^*_{s,o}, \chi_D S_{f,o} \phi_o >}{< \mathbf{n}^*_{s,o}, \bar{\chi} S_{f,o} \phi_o >} \tag{3.57}$$

$$\zeta = \frac{1}{< \mathbf{n}^*_{s,o}, \bar{\chi} S_{f,o} \phi_o >} \tag{3.58}$$

$$\xi_i = \frac{< m^*_{s,o} m_i >}{< \mathbf{n}^*_{s,o}, \bar{\chi} S_{f,o} \phi_o >} \tag{3.59}$$

Eqs. (3.50) and (3.48) may then be written in the form

$$\ell_{eff} \frac{dP}{dt} = (\rho_{gen} - \alpha\beta)P + \alpha \sum_{i=1}^{I} \lambda_i \xi_i + \zeta(1-P) + \rho_{source} \tag{3.60}$$

$$\frac{d\xi_i}{dt} = \beta_i P - \lambda_i \xi_i \tag{3.61}$$

with $P=P_o=1$ and $\xi_i = \beta_i/\lambda_i$ at steady state conditions. The expression for ρ_{gen} was discussed in the previous section.

It is interesting also to note that, with the system approaching criticality, quantity ζ vanishes. Consequently, the third term at the right side of Eq. (3.60) also vanishes (whereas the space distribution of $\mathbf{n}^*_{s,o}$ approaches the standard adjoint flux ϕ^*_o (Gandini, 1997). In this case, Eqs. (3.60) and (3.61) reduce to the homogeneous, standard form of the point kinetics equations. Searching solutions for functions P and ξ_i of the form $e^{-\omega t}$, we may arrive at the expression

$$\rho = \ell_{eff}\omega + \alpha \sum_{i=1}^{I} \frac{\omega \beta_i}{\omega + \lambda_i} \tag{3.62}$$

with

$$\rho = \frac{< \phi^*_o, \{\delta A + [(1-\beta)\chi_P + \beta\chi_D]\delta S_f\}\phi_o >}{< \phi^*_o, \bar{\chi} S_f \phi_o >} \tag{3.63}$$

and with ℓ_{eff} and α given by Eqs. (3.54) and (3.57) with $\mathbf{n}^*_{s,o}$ replaced by ϕ^*_o. The general solution will be then given by the superimposition of the solutions corresponding to the (M+1) roots ω_ℓ.

Eqs. (3.60) and (3.61) may be considered an extension of the point kinetic equation to subcritical systems. Solving Eq. (3.62), with ρ_{gen} given by Eq. (3.63) in place of ρ, and with ℓ_{eff} and α given by Eqs. (3.54) and (3.56), shall give the (M+1) roots ω_i relevant the exponential solutions of the homogeneous equation associated with Eqs. (3.60) and (3.61). As well known, the general solution shall be given by the sum of the solution of the equivalent homogeneous equation and a particular one.

Asymptotically, if after the perturbation the system is still subcritical, a new (relative) power level P_{as} will be reached, given by the expression

$$P_{as} = \frac{\zeta + \rho_{source}}{\zeta - \rho_{gen}} \ , \tag{3.64}$$

which, as expected, increases with ρ_{source} and ρ_{gen}.

Quantity ζ plays the role of a measure of the system subcriticality. To show this, consider first the two subcriticality measures so far generally adopted

$$K_{eff} = \frac{< \phi^*_o, \bar{\chi} S_{f,o} \phi_o >}{< \phi^*_o, \mathbf{s}_{n,o} > + < \phi^*_o, \bar{\chi} S_{f,o} \phi_o >} \tag{3.65}$$

$$K_{source} = \frac{< \mathbf{u}, \bar{\chi} S_{f,o} \phi_o >}{< \mathbf{u}, \mathbf{s}_{n,o} > + < \mathbf{u}, \bar{\chi} S_{f,o} \phi_o >} \tag{3.66}$$

with **u** a unit vector. K_{eff} is associated with the fundamental mode of the neutron. It has relevance for safety studies implying accidents bringing the system to overcritical conditions. K_{source} is a multiplication factor implying the actual flux, in a source driven system generally formed by a superposition of eigenfunctions. It does not take into account the importance of fission and source neutrons with respect to the power. So, taking this importances into account, and recalling that $< \mathbf{n}^*_o s_{n,o} >= 1$, me may define the multiplication coefficient

$$K_{sub} = \frac{< \mathbf{n}^*_{s,o}, \bar{\chi} S_{f,o} \phi_o >}{1 + < \mathbf{n}^*_{s,o}, \bar{\chi} S_{f,o} \phi_o >} \ . \tag{3.67}$$

Quantity ζ then may be written as

$$\zeta = \frac{1 - K_{sub}}{K_{sub}} \ , \tag{3.68}$$

and may be clearly taken as a consistent measure of the distance of the system from criticality.

It was shown (Gandini, 1997) that for K_{sub} approaching unity, function $\mathbf{n}^*_{s,o}$ diverges, its space shape approaching that of the standard adjoint flux. Correspondingly, ρ_{gen} converges to the standard form of reactivity, Eq. (3.63).

We have seen that the quantity ρ_{gen} plays a role analogous to that of the reactivity in the point kinetics equation for critical systems. We may also verify that this quantity, for the same parameter perturbation, gives a decreasing contribution to the power change with the system subcriticality increasing. This is due to the presence of the source-related term $\zeta(1\text{-}P)$ at the right side of Eq.(3.60), where ζ increases with the subcriticality.

As we have seen, the coefficients appearing in Eqs.(3.60) and (3.61) are all physically meaningful. The generalized reactivity, ρ_{gen}, in particular, may be determined by measurement. In fact, as shown in the previous section, it is given by the product of the source-mode generalized reactivity $\rho_{gen,s}$ associated with the source control [cfr. Eq.(3.21)],

$$\rho_{gen,s} = < \mathbf{n}^*_{s,o}, \left\{ \delta A + (1-\beta)\chi_P + \beta\chi_D]\delta S_f \right\} \phi_o > + \frac{\gamma}{W_o} < \delta\Sigma_f, \phi_o >, \qquad (3.69)$$

by the quantity ζ, given by expression (3.58). Since $\rho_{gen,s}$ corresponds to the source strength change necessary to reset the power level after the perturbation, it is clearly a measurable quantity. For what concerns $\zeta \left(\equiv \frac{1-K_{sub}}{K_{sub}} \right)$, this quantity doesn't seem easily amenable to experimental evaluation. It seems easier to consider the quantity $\left(\frac{1-K_{eff}}{K_{eff}} \right)$, obtained by substituting $\mathbf{n}^*_{s,o}$ with the standard adjoint function ϕ^*_o, and then measure it via fundamental mode period measurements. ζ could be then evaluated by multiplying its calculated value by a bias factor, i.e.,

$$\zeta^{exp} = \zeta^{cal} \frac{\left(\frac{1-K_{eff}}{K_{eff}} \right)^{exp}}{\left(\frac{1-K_{eff}}{K_{eff}} \right)^{cal}} . \qquad (3.70)$$

Of course, a similar procedure could be also followed for determining via a bias factor ρ^{exp}_{gen} starting from the measurement of a standard reactivity value ρ^{exp}.

In above expressions we have assumed, for simplicity of presentation, constant values for the delayed neutron fractions β and β_j. In reality these quantities are generally dependent on energy and space, in correspondence to the space distribution of the fuel elements composition and of the neutron energy spectrum. The correct values to be adopted in the above equations are discussed in Appendix B.

Illustrative Example.

Let us consider the simple case of one-group, one precursor, infinite system . In this case Eqs. (3.35) and (3.36) become

$$\frac{1}{v}\frac{d\phi}{dt} = -\Sigma_c\phi + (1-\beta)\nu\Sigma_f\phi + \lambda m + s_n \tag{3.71}$$

$$\frac{dm}{dt} = \beta\nu\Sigma_f\phi - \lambda m \ . \tag{3.71}$$

At unperturbed conditions it is:

$$-\Sigma_{c,o}\phi_o + \nu\Sigma_{f,o}\phi_o + s_{n,o} = 0 \tag{3.72}$$

with solutions

$$\phi_o = \frac{1}{\Sigma_{c,o}}\frac{s_{n,o}}{1-K_o} \tag{3.73}$$

$$m_o = \frac{\beta}{\lambda}\nu\Sigma_f\phi_o = \frac{\beta}{\lambda}\frac{K_o}{1-K_o}s_{n,o} \tag{3.74}$$

The importance function $n^*_{s,o}$ is governed by the equation

$$-\Sigma_{co}n^*_{s,o} + \nu\Sigma_{fo}n^*_{s,o} + \frac{1}{W_o}\gamma\Sigma_{fo} \ = \ 0 \tag{3.75}$$

with the solution

$$n^*_{s,o} = \frac{1}{W_o}\frac{\gamma\Sigma_{fo}}{\Sigma_{co}-\nu\Sigma_{fo}} \equiv \frac{1}{W_o}\frac{\gamma}{\nu}\frac{K_o}{1-K_o} \tag{3.76}$$

As the (reference) system approaches criticality, and then $s_{n,o}$, for the same power, goes to zero, the importance $n^*_{s,o}$ diverges. If, on the contrary, it become increasingly subcritical, it correspondingly reduces, vanishing with $\Sigma_{f,o}$ approaching zero. This is expected recalling the meaning of importance[1].

[1] An importance function (Gandini, 1987) is strictly associated with a response defined in a given space interval (at a limit, at a given time point). To exemplify, with the above one-group, infinite medium, the importance f^* relevant to the power defined at an arbitrary time t' would be governed by the equation:

$$-\frac{df^*}{dt} = -\Sigma_{c,o}f^* + \nu\Sigma_{f,o}f^* + \frac{1}{W_o}\gamma\Sigma_{f,o}\delta(t-t') \tag{a}$$

Consider a perturbation altering the system parameters. The governing equations will result

$$\ell_{\text{eff}} \frac{dP}{dt} = (\rho_{\text{gen}} - \beta)P + \lambda\xi + \frac{1-K_o}{K_o}(1-P) + \rho_{\text{source}} \tag{3.77}$$

$$\frac{d\xi}{dt} = \beta P - \lambda\xi \,. \tag{3.78}$$

If after the perturbation the system is still subcritical, the new power asymptotic level will be

$$P_{as} = \frac{1-K_o + K_o\rho_{\text{source}}}{1-(K_o + K_o\rho_{\text{gen}})} \,. \tag{3.79}$$

As expected, the condition for remaining at subcriticality condition is that $\rho_{\text{gen}} < \frac{1-K_o}{K_o}$.

Assume now the values:

$\ell_{\text{eff}} = 10^{-3}$, λ=0.3 , β=0.007 .

A number of illustrative relevant to different reactivity insertions as shown in Figg. 1 through 5 (showing P vs. sec).

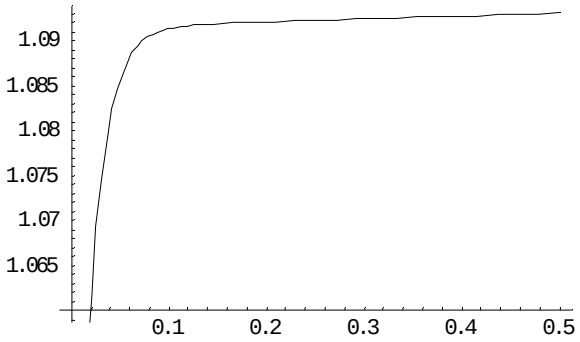

Fig. 1. ρ_{gen} = 0.005 (asymptotic value: P=1.11)

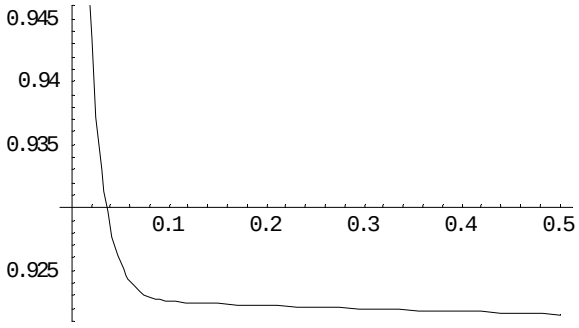

Fig 2. ρ_{gen} = -0.005 (asymptotic value P=0.91)

Integrating from -∞ and t_o^+, recalling that for a subcritical, dissipative system f^* vanishes for t→-∞ and at t > t', and defining the integrated importance

$$n_o^* = \int_{-\infty}^{t'} f^*(t)dt \,, \tag{b}$$

we easily obtain Eq. (3.75). It is also clear that for the system approaching criticality (since the introduction of a neutron at an asymptotic negative time increasingly affects the power value at t') the value n_o^* given by Eq. (b) diverges.

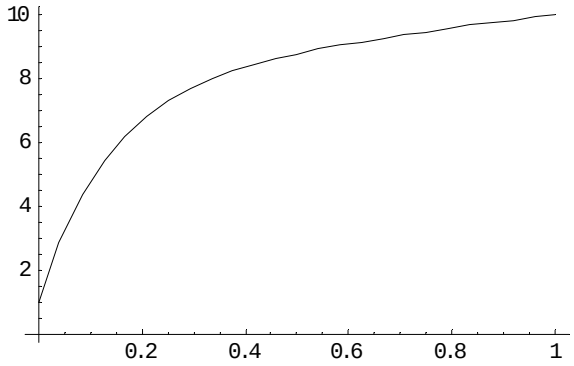

Fig. 3. ρ_{gen} = 0.0526 (critical conditions)

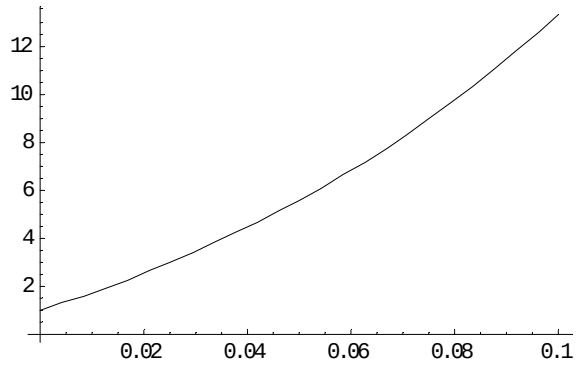

Fig. 4. ρ_{gen} = 0.07 (over prompt critical)

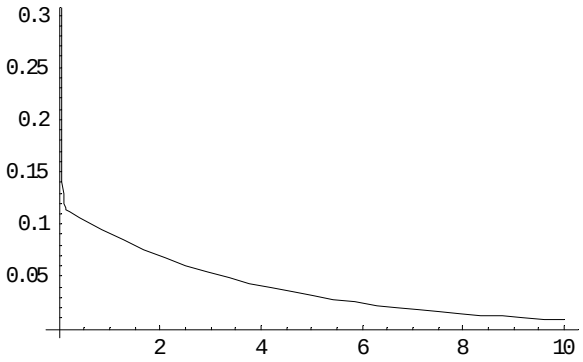

Fig. 5. ρ_{source} = -1 (source removal)

APPENDIX A. Cycle analysis

To the neutron and fuel nuclide densities, represented by vectors **n**(**r**,t) and **c**(**r**,t), respectively, defined in the reactor cycle interval (t_o,t_F), a specified intensive control variable, ρ(t), is associated so that the assigned, overall power history W(t) is maintained. Vector **n** represents the space- and time-dependent neutron density in a multigroup energy form, whereas vector **c** the space- and time-dependent density of the various fuel nuclide species. The intensive, time-dependent, control variable ρ(t) may represent, for instance, the overall control rod bank penetration into the core [not their relative movement, which is generally described by parameters p_k (k=1,2,...)], or the average neutron poison material density. The nonlinear governing equations can then be written formally as

$$\mathbf{m}_{(n)}(\mathbf{n},\mathbf{c},\rho \mid \mathbf{p}) = -\frac{\partial \mathbf{n}}{\partial t} + B\mathbf{n} + \mathbf{s}_n = 0 \qquad (A.1)$$

$$\mathbf{m}_{(c)}(\mathbf{n},\mathbf{c} \mid \mathbf{p}) = -\frac{\partial \mathbf{c}}{\partial t} + E\mathbf{c} + \mathbf{s}_c = 0 \qquad (A.2)$$

$$\mathbf{m}_{(\rho)}(\mathbf{n},\mathbf{c} \mid \mathbf{p}) = <\mathbf{c}, S\mathbf{n}> - W = 0 \qquad (A.3)$$

where B is the neutron diffusion, or transport, matrix operator (depending on **c** and ρ), E the nuclide evolution matrix (depending on **n**), $\mathbf{s}_n$ and $\mathbf{s}_C$ are given source terms[+] , while

$$S = \gamma \begin{vmatrix} \sigma_{f,1}^{1} & \dots & \sigma_{f,G}^{1} \\ \dots & \dots & \dots \\ \sigma_{f,1}^{J} & \dots & \sigma_{f,G}^{J} \end{vmatrix} V , \qquad (A.4)$$

γ being the amount of energy per fission, and $\sigma_{f,g}^{j}$ the microscopic g'th group fission cross-section of the j'th heavy isotope. V is the diagonal neutron velocity matrix. Quantities γ, V, W and $\sigma_{f,g}^{j}$ may be considered generally represented by (or function of) system parameters p_k. Source terms $\mathbf{s}_n$ and $\mathbf{s}_c$ are also parameter dependent.

In quasi-static problems, as those of interest here, the derivative $\frac{\partial \mathbf{n}}{\partial t}$ is negligible.

If we introduce the field

$$\mathbf{f}(\mathbf{r},t) = \begin{vmatrix} \mathbf{n} \\ \mathbf{c} \\ \rho \end{vmatrix} \qquad (A.5)$$

the system of Eqs. (A.1), (A.2) and (A.3) may be represented in the compact symbolic form, Eq. (2.1), and the HGPT methodology described above applied.

Consider a functional

$$Q = \int_{t_o}^{t_F} < \begin{vmatrix} \mathbf{s}_n^+ & \mathbf{s}_c^+ & \mathbf{s}_\rho^+ \end{vmatrix} \mathbf{f} > \begin{vmatrix} \mathbf{n}(\mathbf{r},t) \\ \mathbf{c}(\mathbf{r},t) \\ \rho(t) \end{vmatrix} dt . \qquad (A.6)$$

Q may represent, for instance, the amount of a given nuclide built up at time t_F [in this case $\mathbf{s}_n^+ = 0$, $\mathbf{s}_\rho^+ = 0$ and $\mathbf{s}_c^+$ includes a delta function $\delta(t-t_F)$], or the fluence at a specific point $\bar{r}$ [in this case $\mathbf{s}_c^+ = 0$, $\mathbf{s}_\rho^+ = 0$ and $\mathbf{s}_n^+$ includes a delta function $\delta(r-\bar{r})$. The importance function

$$\mathbf{f}^*(\mathbf{r},t) = \begin{vmatrix} \mathbf{n}^*(\mathbf{r},t) & \mathbf{c}^*(\mathbf{r},t) & \rho^*(t) \end{vmatrix}^T \qquad (A.7)$$

can then be defined, and results governed by Eq. (2.3), with H^* and $\mathbf{h}^+$ given by expressions:

[+] $\mathbf{s}_n$ is generally assumed zero during burnup, except a delta-like source at t_o to represent initial conditions (usually considered at steady state), whereas $\mathbf{s}_C$ is generally given by a sum of delta functions defined at t_o and at given times to account for fuel feed and shuffling operations.

$$H^* = \begin{vmatrix} (\frac{\partial}{\partial t} + B^*) & \Omega_c^* & S^T\mathbf{c} \\ \Omega_n^* & (\frac{\partial}{\partial t} + E^T) & S\mathbf{n} \\ < \mathbf{n}, \left(\frac{\partial B}{\partial \rho}\right)^* (\cdot) > & 0 & 0 \end{vmatrix} \qquad \text{(A.8)}$$

$$\mathbf{h}^+ = \begin{vmatrix} \mathbf{s}_n^+ \\ \mathbf{s}_c^+ \\ s_\rho^+ \end{vmatrix} \qquad \text{(A.9)}$$

Ω_c^* and Ω_n^* being operators adjoint of Ω_C [$= \frac{\bar{\partial}(E\mathbf{c})}{\partial \mathbf{n}}$] and Ω_n [$= \frac{\bar{\partial}(B\mathbf{n})}{\partial \mathbf{c}}$], respectively.

The equation relevant to function ρ^* corresponds to a relationship between $\mathbf{n}^*$ and $\mathbf{n}$, i.e., .

$$< \mathbf{n}, \frac{\partial B^*}{\partial \rho}\mathbf{n}^* >= \mathbf{s}_\rho^+ \qquad \text{(A.10)}$$

In case $\mathbf{s}_\rho^+$=0, Eq. (A.4) corresponds to an orthogonality relationship.

To solve the equations relevant to $\mathbf{n}^*$ and $\mathbf{c}^*$ different resolution recurrent schemes may be considered, starting from the 'final' time t_F and proceeding backward, along with the same time discretisation adopted in the forward reference calculation.

It can be shown (Gandini, 1987) that, at quasi static conditions, the equations to be solved reduce to the types:

$$B^*\mathbf{n}^* + \mathbf{h}_n^+ = 0 \qquad \text{(A.11)}$$

$$-\frac{\partial \mathbf{c}^*}{\partial t} = E^T\mathbf{c}^* + \mathbf{h}_c^+ \qquad \text{(A.12)}$$

where $\mathbf{h}_n^+$ and $\mathbf{h}_c^+$ correspond to known source terms determined during the recurrent calculation procedure. Therefore, existing, well established codes can be used for their solution.

The sensitivity coefficient $\frac{dQ}{dp_k}$ with respect to a given parameter p_k may then be obtained from Eq. (2.4), with vector $\mathbf{m}$ made of components $\mathbf{m}_{(n)}$, $\mathbf{m}_{(c)}$ and $m_{(\rho)}$ defined in Eqs. (A.1), (A.2) and (A.3), respectively.

A general problem we are faced with is the following: how does the control criticality reset (ρ) strategy affect the sensitivity analysis results? To answer this question, let us consider Eq. (A.11) governing $\mathbf{n}^*$. We note that, given a particular solution $\mathbf{n}^*_{part}$, the general one may be written as

$$\mathbf{n}^* = \mathbf{n}^*_{part} + a\phi^* \qquad (A.13)$$

where a is an arbitrary coefficient and ϕ^* the conventional adjoint function obeying the homogeneous equation

$$B^*\phi^* = 0 \quad . \qquad (A.14)$$

Once a solution $\mathbf{n}^*_{part}$ has been obtained, the solution desired can then be derived by proper filtering from the fundamental mode, i.e., it will be given by Eq. (A.13), with coefficient a determined by imposing condition (A.10). Assuming $\mathbf{s}^+_\rho$=0, we shall have

$$\mathbf{n}^* = \mathbf{n}^*_{part} - \frac{< \mathbf{n}^*_{part}, \frac{\partial B}{\partial \rho}\mathbf{n} >}{< \phi^*, \frac{\partial B}{\partial \rho}\mathbf{n} >} \phi^* \quad . \qquad (A.15)$$

The dependence of the importance function $\mathbf{n}^*$ on the control mode adopted is evident.

When calculating the sensitivity coefficient of a response Q with respect to a given parameter p_k (or its change δQ with respect to parameter alterations δp_k), the filtering of the importance function as shown in Eq. (A.15) corresponds to *implicitly* accounting for the ρ-mode control reset of the criticality (in the following we shall refer to it simply as ρ-mode reset).

The above result may have important implications, in the sense that in many circumstances, prior to a sensitivity study, it may be necessary to consider the proper reactivity control mode to be adopted. On the other hand, within many existing codes used with the HGPT methodology, the fictitious "λ-mode" reset control is implicitly assumed, i.e., that related to the coefficient (eigenvalue) λ multiplying the fission source term (F**n**) in the Boltzmann (or diffusion) equation. In this circumstance expression (A.15) for the importance $\mathbf{n}^*$ will result, recalling that in this case $\frac{\partial B}{\partial \lambda} = F$,

$$\mathbf{n}^* = \mathbf{n}^*_{part} - \frac{< \mathbf{n}^*_{part}, F\mathbf{n} >}{< \phi^*, F\mathbf{n} >} \phi^* \quad . \qquad (A.16)$$

Using this λ-mode filtering, rather than the correct ρ-mode one, may lead to erroneous results.

Consider, for instance, the case of a sensitivity analysis with respect to core breeding, or conversion ratio, a quantity clearly dependent on the neutron energy spectrum. Assuming that the reactivity compensation corresponding to the change of a system parameter (for instance, the initial fuel enrichment) is effected, as it may very well be the case for a thermal reactor, by an alteration of

the average (boron) poison concentration in the coolant, the correct choice of the control mode reset would clearly have the effect of hardening (if boron is added), or softening (if boron is subtracted) the neutron spectrum. Instead, if a λ-mode reset would have been implicitly adopted (as is often done with existing codes), no significant neutron energy shift would have been taken into account, and, consequently, an erroneous sensitivity coefficient would result.

It is also true that in principle one could calculate separately the amount of control poison (referring to the above example) to reset the criticality and consider the overall parameter plus control change along with the λ-mode methodology. But this would imply a reactivity reset calculation to be performed for each parameter considered. On the other hand, the correct fundamental ρ-mode filtering may be a quite straightforward procedure. In fact, it can be effected "a posteriori" adopting expression (A.15) in which $\mathbf{n}^{*}_{\text{part}}$ would correspond to a preliminary λ-mode calculation with an existing code.

APPENDIX B. Beta effective

In the expressions derived in Section 3 we have assumed, for simplicity of presentation, constant values for the delayed neutron fractions β and β_i. In reality these quantities are generally dependent on energy and space, in correspondence to the space distribution of the fuel elements composition and of the neutron energy spectrum. To obtain the effective quantities to be inserted in the point kinetic equations, we should consider, in place of Eqs. (3.37) and (3.38), the following ones, indicating with c_j (j=1,2, ...,J) the density of the j'th heavy isotope,

$$V^{-1}\frac{d\phi}{dt} = A\phi + \sum_{j=1}^{J}\sum_{i=1}^{I} c_j \chi_P^j (U - B_{i,j}) S_j \phi + \sum_{j=1}^{J} c_j X_D^j \Lambda \mathbf{m}^j + \mathbf{s}_n \qquad \text{(B.1)}$$

$$\frac{d\mathbf{m}^j}{dt} = B_j S_j \phi - \Lambda \mathbf{m}^j \qquad \text{(B.2)}$$

where we also accounted for a general dependence of the prompt and delayed fission neutron spectrum on the heavy nuclide species undergoing fission, and where, with evident notation,

$$U = \begin{vmatrix} 1 & \dots & 1 \\ \dots & \dots & \dots \\ 1 & \dots & 1 \end{vmatrix}_{(GxG)} \quad B_{i,j} = \begin{vmatrix} \beta^j_{i,1} & \dots & \beta^j_{i,G} \\ \dots & \dots & \dots \\ \beta^j_{i,1} & \dots & \beta^j_{i,G} \end{vmatrix}_{(GxG)} \quad B_j = \begin{vmatrix} \beta^j_{1,1} & \dots & \beta^j_{1,G} \\ \dots & \dots & \dots \\ \beta^j_{I,1} & \dots & \beta^j_{I,G} \end{vmatrix}_{(IxG)} \quad S_j = \begin{vmatrix} \nu\sigma^j_{f,1} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \nu\sigma^j_{f,G} \end{vmatrix} .$$

Eqs. (B.1) and (B.2)) could be written in the form of Eqs. (3.37) and (3.38) if we replace β_i and β by the expressions, assuming for the i'th delayed neutron fraction a general dependence on nuclide species and incident neutron energy,

$$\beta_{i,eff} = \frac{\sum_{g=1}^{G}\sum_{j=1}^{J} < n^{*}_{s,o,g} c_j \chi^{j}_{D,g} \beta^{j}_{i,g} \nu\sigma^{j}_{f,g} \phi_g >}{\sum_{g=1}^{G}\sum_{j=1}^{J}\sum_{i=1}^{I} < n^{*}_{s,o,g} c_j \chi^{j}_{D,g} \nu\sigma^{j}_{f,g} \phi_g >} \qquad \text{(B.3)}$$

$$\beta_{eff} = \sum_{i=1}^{I} \beta_{i,eff} \quad . \qquad \text{(B.4)}$$

In place of Eqs. (3.60) and (3.61) we would then have:

$$\ell_{eff} \frac{dP}{dt} = (\rho_{gen} - \alpha\beta_{eff})P + \alpha \sum_{i=1}^{I} \lambda_i \xi_i + \zeta(1-P) + \rho_{source} \qquad \text{(A.5)}$$

$$\frac{d\xi_i}{dt} = \beta_{i,eff} P - \lambda_i \xi_i \ . \qquad \text{(A.6)}$$

REFERENCES

Cacuci D.G., Oblow E.M., Marable J.H., Weber C.F. (1880) *Nucl.Sci.Eng.*,**75**, 88.

Gandini, A. (1967) *J. Nucl.Energy*, **21**, 755.

Gandini A. (1969) *Nucl.Sci.Eng.*, **35**,141.

Gandini A. (1976) *Nucl.Sci.Eng.*, **59**,60.

Gandini A. (1981) *Nucl.Sci.Eng.*, **77**,316.

Gandini A. (1983)*Trans.Am.Nucl.Soc.*, **45**, 325.

Gandini A. (1987) "Generalized Perturbation Theory Methods. A Heuristic Approach", in *Advances in Nuclear Sci. and Techn.*,Vol 19, J. Lewins and M. Becker Eds., Plenum Press, New York, p 205.

A. Gandini (1987) *Ann. Nucl. Energy*, **14**, 273.

A. Gandini (1988) *Ann. Nucl. Energy*, **15**, 327.

Gandini A. (1997) *Ann. Nucl. Energy*, **24**, 1241, and "A Sensitivity Approach to the Analysis of Source Driven Systems", Meeting on Feasibility and Motivation for Hybrid Concepts for Nuclear Energy Generation and Transmutation, Madrid, Sept. '97 [Document IAEA -TC-903.3, p. 377].

Greenspan E.M. (1975) *Nucl.Sci. Eng.*, **57**, 250 .

Harris D.R., Becker M. (1976) *Trans. Am. Soc.*, **23**, 534.

Hurwitz H. (1949) Nucleonics, 5, No.1, 62.

Lewins J. (1965) *Importance, the Adjoint Function*, Pergamon Press, Oxford.

Oblow E.M. (1976) *Nucl.Sci.Eng.*, **59**, 187.

Pomraning G. (1967) *J. Math.Phys.*, **8**, 149.

Soodak H. (1948) *The Science and Engineering of Nuclear Power*, United Nations, New York (1948).

Stacey, W.M. Jr. (1974), *Variational Methods in Nuclear Reactor Physics*, Acad. Press, New York.

Usachev L.N. (1955) 1st ICPUE-UN, 5, 503

Usachev L.N. (1963) *Atomnaya Energiya*, **15**,472.
Wigner, E.P. (1945) Effect of Small Perturbations on Pile Period, Chicago Report CP-G-3048.
Williams M.L.(1979) *Nucl.Sci.Eng*., **70**, 20.